Some asymptotic limits for solutions of Burgers equation

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§1 – Introduction. In this paper, we compute the limits

$$\gamma_p \;=\; \lim_{t \to \infty} \; t^{\frac{1}{2}\left(1 - \frac{1}{p}\right)} \, \| \, u(\cdot,t) \, \|_p, \qquad 1 \leq p \leq \infty \tag{1}$$

for solutions $u(\cdot,t)$ of the equation

$$u_t \;+\; a\,u_x \;+\; b\,u\,u_x \;=\; c\,u_{xx} \tag{2}$$

satisfying the Cauchy condition

$$u(x,0) \;=\; u_0(x), \qquad u_0 \in L^1(\mathbf{R}), \tag{3}$$

that is, $\| \, u(\cdot,t) \,-\, u_0 \, \|_1 \;\to\; 0$ as $t \;\to\; 0$, $t \,>\, 0$. Here, $\| \, u(\cdot,t) \, \|_p$ denotes the L^p norm of $u(\cdot,t)$ as a function of x for fixed t, i.e.,

$$\| \, u(\cdot,t) \, \|_p \;=\; \left(\int_{-\infty}^{+\infty} | \, u(x,t) \, |^p \, dx \right)^{1/p} \tag{4}$$

if $1 \leq p < \infty$, and

$$\| \, u(\cdot,t) \, \|_\infty \;=\; \sup_{x \in \mathbf{R}} \; | \, u(x,t) \, | \tag{5}$$

for $p = \infty$. In equation (2) above, a, b, c are real constants, with $c > 0$. When $b = 0$ we have the familiar heat equation; our main concern is the case $b \neq 0$, the so-called Burgers equation [1], [3]. Using the Hopf-Cole transformation [4], [5], it is well known that the solution in this case is given by

$$u(x,t) \;=\; \frac{1}{\sqrt{4\,\pi\,c\,t}} \, \frac{1}{\varphi(x,t)} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y-a\,t)^2}{4\,c\,t}} \, \varphi_0(y) \, u_0(y) \; dy, \tag{6}$$

with

$$\varphi(x,t) \;=\; \frac{1}{\sqrt{4\,\pi\,c\,t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y-a\,t)^2}{4\,c\,t}} \, \varphi_0(y) \; dy, \tag{7}$$

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where $\varphi_0 \in L^\infty(\mathbf{R})$ is the Hopf-Cole transform of the initial state u_0, i.e.,

$$\varphi_0(x) \;=\; \mathsf{e}^{-\frac{b}{2c}\int_{-\infty}^{x} u_0(\xi)\,d\xi}. \tag{8}$$

From these expressions, we easily get that, for every $1 \leq p \leq \infty$, one has

$$\| u(\cdot,t) \|_p \;\leq\; C\, t^{-\frac{1}{2}(1-\frac{1}{p})} \tag{9}$$

for some constant $C > 0$ which depends on the magnitude of $\| u_0 \|_1$, but it is not immediately evident how to compute the limits γ_p above. Denoting by m the total mass of the solution, i.e.,

$$m \;=\; \int_{-\infty}^{+\infty} u_0(x)\,dx, \tag{10}$$

we will show that

$$\gamma_p \;=\; \frac{|\,m\,|}{\sqrt{4\pi c}}\,(4c)^{\frac{1}{2p}} \left| \frac{2c}{bm}\left(1 - \mathsf{e}^{-\frac{bm}{2c}}\right)\right| \| \mathcal{F} \|_p \tag{11}$$

with $\mathcal{F} \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ being a function which depends on the parameters b, c, m above, given by

$$\mathcal{F}(x) \;=\; \frac{\mathsf{e}^{-x^2}}{\mu \,-\, h\,\mathsf{erf}\,(x)}, \tag{12}$$

where $\mathsf{erf}\,(x)$ is the error function

$$\mathsf{erf}\,(x) \;=\; \frac{1}{\sqrt{\pi}} \int_0^x \mathsf{e}^{-\xi^2}\,d\xi \tag{13}$$

and μ, h are given by

$$\mu \;=\; \frac{1 + \mathsf{e}^{-\frac{bm}{2c}}}{2}, \qquad h \;=\; |\; 1 - \mathsf{e}^{-\frac{bm}{2c}}\;|. \tag{14}$$

When $p = 1$, (1), (11) become

$$\lim_{t\to\infty} \| u(\cdot,t) \|_1 \;=\; |\,m\,|. \tag{15}$$

In the case of heat equation, i.e., $b = 0$, the corresponding results for γ_p are given by the limiting values of the right-hand-side in (11) as $b \to 0$, i.e.,

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \| u(\cdot,t) \|_p \;=\; \frac{|\,m\,|}{\sqrt{4\pi c}} \left(\frac{4\pi c}{p}\right)^{\frac{1}{2p}} \tag{16}$$

for every $1 \leq p \leq \infty$. This case is easier and is briefly considered in Section 2 below.

The more interesting case when $b \neq 0$ is then taken up in more detail in Section 3. It is also shown in Section 3 that, as it should be expected, the equations in the class (2) are not asymptotically equivalent to one another: if $u, \hat{u}$ are solutions of

$$u_t + a\,u_x + b\,u\,u_x = c\,u_{xx} \tag{17}$$

and

$$\hat{u}_t + \hat{a}\,\hat{u}_x + \hat{b}\,\hat{u}\,\hat{u}_x = \hat{c}\,\hat{u}_{xx} \tag{18}$$

corresponding to the same initial profile $u_0 \in L^1(\mathbf{R})$ with some nonzero mass, and $(a, b, c) \neq (\hat{a}, \hat{b}, \hat{c})$, then, for every $1 \leq p \leq \infty$, there exist positive constants c_p, T_p such that

$$\| u(\cdot,t) - \hat{u}(\cdot,t) \|_p \;\geq\; c_p\, t^{-\frac{1}{2}\left(1 - \frac{1}{p}\right)} \tag{19}$$

for all $t \geq T_p$, so that $\| u(\cdot,t) - \hat{u}(\cdot,t) \|_p$ decays at exactly the same speed as each term $\| u(\cdot,t) \|_p$, $\| \hat{u}(\cdot,t) \|_p$ on its own.

§2 – The case $b = 0$. Before we derive the results for the Burgers equation, it will be convenient to consider briefly the simple case of heat equation. Clearly, it is sufficient to examine the case when $a = 0$, so that we assume in this section that $u(\cdot,t) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ is the solution of the initial value problem

$$u_t = c\,u_{xx} \tag{20}$$

$$u(x,0) = u_0(x) \tag{21}$$

where $c > 0$ is constant and $u_0 \in L^1(\mathbf{R})$. It is well known that $u(x,t)$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi c t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y)^2}{4ct}}\, u_0(y)\, dy, \tag{22}$$

so that it satisfies, for every $1 \leq p \leq \infty$,

$$\| u(\cdot,t) \|_p = O(1)\, t^{-\frac{1}{2}\left(1 - \frac{1}{p}\right)}. \tag{23}$$

A more subtle result which will be important throughout the analysis is given in the following lemma.

Lemma 1
Let $u_0 \in L^1(\mathbf{R})$ be such that $\int_{-\infty}^{+\infty} u_0(x)\, dx = 0$. Then, for every $1 \leq p \leq \infty$, one has

$$\lim_{t\to\infty} t^{-\frac{1}{2}\left(1 - \frac{1}{p}\right)} \| u(\cdot,t) \|_p = 0. \tag{24}$$

$\square$ We will first show that $\lim_{t\to\infty} \| u(\cdot,t) \|_1 = 0$. This has been shown for linear equations more general than (20) in [2], [6], but for convenience we will give a direct derivation below. Given $\varepsilon > 0$, let $A > 0$ be chosen such that

$$\int_{|y|\geq A} | u_0(y) | \, dy \;\leq\; \varepsilon,$$

so that, from (22),

$$\| u(\cdot,t) \|_1 \;\leq\; \varepsilon \,+\, \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi c t}} \,\Big|\, \int_{|y|\leq A} e^{-\frac{(x-y)^2}{4ct}} \, u_0(y) \, dy \,\Big|\, dx$$

$$= \;\varepsilon \,+\, \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \Big|\, \int_{|y|\leq A} e^{-\left(\xi - \frac{y}{\sqrt{4ct}}\right)^2} \, u_0(y) \, dy \,\Big|\, d\xi.$$

Letting $t \to \infty$, we then get

$$\limsup_{t\to\infty} \| u(\cdot,t) \|_1 \;\leq\; \varepsilon \,+\, \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} \,\Big|\, \int_{|y|\leq A} u_0(y) \, dy \,\Big|\, d\xi$$

$$\leq\; \varepsilon \,+\, \Big|\, \int_{|y|\leq A} u_0(y) \, dy \,\Big| \;\leq\; 2\,\varepsilon,$$

where we have used that $\int_{-\infty}^{+\infty} u_0(y) \, dy = 0$. Since $\varepsilon > 0$ is arbitrary, this gives

$$\lim_{t\to\infty} \| u(\cdot,t) \|_1 \;=\; 0,$$

which concludes the case $p = 1$. Now, given $1 < p < \infty$, we have, using (23),

$$t^{\frac{1}{2}\left(1 - \frac{1}{p}\right)} \| u(\cdot,t) \|_p \;\leq\; \| u(\cdot,t) \|_1^{\frac{1}{p}} \left(t^{\frac{1}{2}} \| u(\cdot,t) \|_\infty \right)^{1 - \frac{1}{p}} \;\leq\; C \, \| u(\cdot,t) \|_1^{\frac{1}{p}}$$

for some constant $C > 0$, so that

$$\lim_{t\to\infty} t^{\frac{1}{2}\left(1 - \frac{1}{p}\right)} \| u(\cdot,t) \|_p \;=\; 0$$

from the previous case. Finally, we consider the case $p = \infty$: from (22), it readily follows that

$$\| u_x(\cdot,t) \|_2 \;=\; O(1) \, t^{-\frac{3}{4}},$$

so that

$$t^{\frac{1}{2}} \| u(\cdot,t) \|_\infty = O(1) \, t^{\frac{1}{2}} \| u(\cdot,t) \|_2^{\frac{1}{2}} \, \| u_x(\cdot,t) \|_2^{\frac{1}{2}} \;=\; O(1) \left(t^{\frac{1}{2}} \| u(\cdot,t) \|_2 \right)^{\frac{1}{2}},$$

which gives the result from the case $p = 2$ already considered. $\square$

Using the estimates above, we can easily obtain the limits γ_p for (20), (21), as shown next.

Theorem 1
Given $u_0 \in L^1(\mathbf{R})$*, the solution* $u(x,t)$ *of* (20), (21) *satisfies*

$$(i) \qquad \lim_{t\to\infty} \| u(\cdot,t) \|_1 \;=\; |\, m \,|,$$

$$(ii) \qquad \lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \| u(\cdot,t) \|_p \;=\; \frac{|\, m\,|}{\sqrt{4\pi c}} \left(\frac{4\pi c}{p} \right)^{\frac{1}{2p}},$$

and

$$(iii) \qquad \lim_{t\to\infty} t^{\frac{1}{2}} \| u(\cdot,t) \|_\infty \;=\; \frac{|\, m\,|}{\sqrt{4\pi c}},$$

where $m = \int_{-\infty}^{+\infty} u_0(x)\, dx.$

□ In fact, this can be readily established for

$$\tilde{u}(x,t) \;=\; \frac{m}{\sqrt{4\pi c t}} \int_0^1 \mathrm{e}^{-\frac{(x-y)^2}{4ct}}\, dy,$$

i.e., the solution of (20), (21) corresponding to the initial profile $\tilde{u}_0(x) = m\,\chi_{[0,1]}(x)$, where $\chi_{[0,1]}$ denotes the characteristic function of the interval $[0,1]$. The result then follows for an arbitrary $u_0 \in L^1(\mathbf{R})$ with the same mass m, because, from Lemma 1, one has

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \| u(\cdot,t) - \tilde{u}(\cdot,t) \|_p \;=\; 0. \qquad \square$$

§3 – The case $b \neq 0$. In this section we extend the analysis above to the more interesting case of Burgers equation. As before, we assume without loss of generality $a = 0$, and let $u(\cdot,t)$ be the solution of the Cauchy problem

$$\text{(25a)} \qquad u_t + b\,u\,u_x \;=\; c\,u_{xx},$$

$$\text{(25b)} \qquad u(x,0) \;=\; u_0(x),$$

where $b \neq 0$, $c > 0$, and $u_0 \in L^1(\mathbf{R})$. Using the Hopf-Cole transformation [3], [4]

$$\text{(26)} \qquad \varphi(x,t) \;=\; \mathrm{e}^{-\frac{b}{2c}\int_{-\infty}^{x} u(\xi,t)\,d\xi},$$

we have that $\varphi(\cdot,t)$ satisfies

(27a) $$\varphi_t \;=\; c\,\varphi_{xx}$$

(27b) $$\varphi(x,0) \;=\; \varphi_0(x) \;\equiv\; \mathsf{e}^{\,-\,\frac{b}{2\,c}\int_{-\infty}^{x} u_0(\xi)\,d\xi}$$

with $u(x,t)$ given by

(28) $$u(x,t) \;=\; -\,\frac{2\,c}{b}\,\frac{\varphi_x(x,t)}{\varphi(x,t)},$$

that is,

(29) $$u(x,t) \;=\; \frac{1}{\sqrt{4\,\pi\,c\,t}}\,\frac{1}{\varphi(x,t)}\int_{-\infty}^{+\infty} \mathsf{e}^{\,-\,\frac{(x-y)^2}{4\,c\,t}}\,\varphi_0(y)\,u_0(y)\,dy.$$

It follows from this expression that $u(\cdot,t) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ for all $t > 0$, with

(30) $$\|\,u(\cdot,t)\,\|_p \;=\; O(1)\,t^{\,-\frac{1}{2}\left(1\,-\,\frac{1}{p}\right)}$$

for every $1 \le p \le \infty$. Moreover, when u_0 has zero mass, we readily get the following estimate from Lemma 1.

Lemma 2
Let $u_0 \in L^1(\mathbf{R})$ be such that $\int_{-\infty}^{+\infty} u_0(x)\,dx \;=\; 0$. Then, for every $1 \le p \le \infty$, one has

(31) $$\lim_{t\to\infty}\; t^{\,-\frac{1}{2}\left(1\,-\,\frac{1}{p}\right)}\;\|\,u(\cdot,t)\|_p \;=\; 0.$$

$\square$ In fact, from (27) we see that φ_x satisfies the conditions of Lemma 1, so that, for every $1 \le p \le \infty$,

$$\lim_{t\to\infty}\; t^{\,\frac{1}{2}\left(1\,-\,\frac{1}{p}\right)}\;\|\,\varphi_x(\cdot,t)\,\|_p \;=\; 0.$$

Since $1/\varphi(\cdot,t)$ is uniformly bounded, the same is true of $u(\cdot,t)$ in view of (28). $\square$

Another fundamental consequence of Lemma 1 is given next.

Lemma 3
Let $u_0, v_0 \in L^1(\mathbf{R})$ be such that

(32) $$\int_{-\infty}^{+\infty} u_0(x)\;dx \;=\; \int_{-\infty}^{+\infty} v_0(x)\;dx,$$

and let $u(\cdot,t)$, $v(\cdot,t)$ be the solutions of (25) corresponding to the initial profiles u_0, v_0, respectively. Then, for every $1 \le p \le \infty$, one has

(33) $$\lim_{t\to\infty}\; t^{\,\frac{1}{2}\left(1\,-\,\frac{1}{p}\right)}\;\|\,u(\cdot,t)\;-\;v(\cdot,t)\,\|_p \;=\; 0.$$

$\square$ Letting $\varphi(\cdot,t)$, $\psi(\cdot,t)$ be the Hopf-Cole transforms of $u(\cdot,t)$, $v(\cdot,t)$, respectively, i.e.,

$$\varphi(x,t) \;=\; \mathsf{e}^{-\frac{b}{2c}\int_{-\infty}^{x} u(\xi,t)\,d\xi}, \qquad \psi(x,t) \;=\; \mathsf{e}^{-\frac{b}{2c}\int_{-\infty}^{x} v(\xi,t)\,d\xi},$$

and setting $\omega = \varphi_x - \psi_x$, we have that $\omega(\cdot,t)$ has zero mass and satisfies $\omega_t \;=\; c\,\omega_{xx}$, so that, from Lemma 1,

$$\lim_{t\to\infty} t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \,\|\, \varphi_x(\cdot,t) \,-\, \psi_x(\cdot,t) \,\|_p \;=\; 0$$

for every $1 \le p \le \infty$, i.e.,

$$\lim_{t\to\infty} t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \,\|\, \varphi(\cdot,t)\,u(\cdot,t) \,-\, \psi(\cdot,t)\,v(\cdot,t) \,\|_p \;=\; 0.$$

Since $1/\varphi(\cdot,t)$, $1/\psi(\cdot,t)$ are uniformly bounded and

$$\lim_{t\to\infty} \,\|\, \varphi(\cdot,t) \,-\, \psi(\cdot,t) \,\|_\infty \;=\; 0,$$

we get the result. $\square$

We are now in position to compute the limits γ_p for an arbitrary u_0 in $L^1(\mathbf{R})$.

Theorem 2
Given $u_0 \in L^1(\mathbf{R})$, *the solution* $u(\cdot,t)$ *of* (25) *satisfies*

(i) $$\lim_{t\to\infty} \,\|\, u(\cdot,t) \,\|_1 \;=\; |\, m \,|,$$

(ii) $$\lim_{t\to\infty} t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \,\|\, u(\cdot,t) \,\|_p \;=\; \frac{|\, m \,|}{\sqrt{4\pi c}}\,(4c)^{\frac{1}{2p}} \left|\, \frac{2c}{bm}\left(1 \,-\, \mathsf{e}^{-\frac{bm}{2c}}\right) \right| \,\|\, \mathcal{F} \,\|_p$$

and

(iii) $$\lim_{t\to\infty} t^{\frac{1}{2}} \,\|\, u(\cdot,t) \,\|_\infty \;=\; \frac{|\, m \,|}{\sqrt{4\pi c}} \left|\, \frac{2c}{bm}\left(1 \,-\, \mathsf{e}^{-\frac{bm}{2c}}\right) \right| \,\|\, \mathcal{F} \,\|_\infty,$$

where $\mathcal{F} \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ *is given in* $(12) - (15)$, *and* $m \;=\; \int_{-\infty}^{+\infty} u_0(x)\,dx$.

$\square$ Because of Lemma 3, it is sufficient to show the result for the particular initial state $u_0 = m\,\chi_{[0,1]}$, in which case $u(\cdot,t)$ is given by

$$u(x,t) \;=\; \frac{m}{\sqrt{4\pi ct}}\,\frac{1}{\varphi(x,t)} \int_0^1 \mathsf{e}^{-\frac{(x-y-at)^2}{4ct}}\,\varphi_0(y)\,dy, \tag{34}$$

where

$$\varphi(x,t) \;=\; \frac{1}{\sqrt{4\pi c t}} \int_{-\infty}^{+\infty} \mathsf{e}^{-\frac{(x-y-at)^2}{4ct}} \varphi_0(y)\, dy, \tag{35}$$

with

$$\varphi_0(x) \;=\; \mathsf{e}^{-\frac{b}{2c}\int_{-\infty}^{x} u_0(\xi)\, d\xi}, \tag{36}$$

see (6), (7), (8). In particular, for any $t > 0$,

$$\| u(\cdot,t) \|_1 \;=\; \left| \int_{-\infty}^{+\infty} u(x,t)\, dx \right| \;=\; \left| \int_{-\infty}^{+\infty} u_0(x)\, dx \right| \;=\; |m|,$$

which shows (i). To get (ii), (iii), we introduce

$$\mathcal{H}_0(x) \;=\; \begin{cases} 1, & x < \alpha \\ \mathsf{e}^{-\frac{bm}{2c}}, & x > \alpha \end{cases} \tag{37}$$

where $\alpha > 0$ is chosen so that

$$\int_{-\infty}^{+\infty} (\mathcal{H}_0(x) - \varphi_0(x))\, dx \;=\; 0, \tag{38}$$

i.e.,

$$\alpha + (1-\alpha)\,\mathsf{e}^{-\frac{bm}{2c}} \;=\; \frac{2c}{bm}\left(1 - \mathsf{e}^{-\frac{bm}{2c}}\right), \tag{39}$$

as illustrated in the picture below.

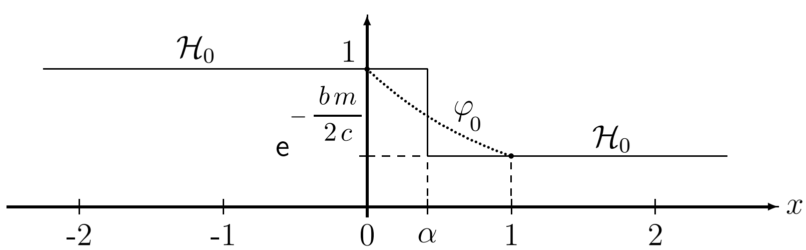

Figure 1: $\mathcal{H}_0$ and φ_0

Setting

$$\mathcal{H}(x,t) \;=\; \frac{1}{\sqrt{4\pi c t}} \int_{-\infty}^{+\infty} \mathsf{e}^{-\frac{(x-y)^2}{4ct}} \mathcal{H}_0(y)\, dy, \tag{40}$$

we have, from (38) and Lemma 1,

$$\lim_{t\to\infty} t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \| \mathcal{H}(\cdot,t) - \varphi(\cdot,t) \|_p \;=\; 0 \tag{41}$$

for every $1 \le p \le \infty$, so that

(42)
$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \| u(\cdot,t) - \omega(\cdot,t) \|_p = 0,$$

where $\omega(\cdot,t)$ is defined by

(43)
$$\omega(x,t) = \frac{m}{\sqrt{4\pi c t}} \frac{1}{\mathcal{H}(x,t)} \int_0^1 \mathrm{e}^{-\frac{(x-y-at)^2}{4ct}} \mathcal{H}_0(y)\, dy,$$

and $\mathcal{H}(\cdot,t)$ is given in (40) above, that is,

(44)
$$\mathcal{H}(x,t) = \mu - \sigma h \,\mathsf{erf}\Big(\frac{x-\alpha}{\sqrt{4ct}} \Big),$$

where σ is the sign of the product bm (i.e., $\sigma = 1$ if $bm > 0$, $\sigma = -1$ otherwise) and μ, h, $\mathsf{erf}(x)$ are given in (13) – (15). We will now derive (ii), for $1 \le p < \infty$: given $\xi \in \mathbf{R}$, we have

(45)
$$\omega(\alpha + \xi\sqrt{4ct}\,, t) = \frac{m}{\sqrt{4\pi c t}} \frac{1}{\mu - \sigma h \,\mathsf{erf}(\xi)} \int_0^1 \mathrm{e}^{-\left(\xi + \frac{\alpha - y}{\sqrt{4ct}}\right)^2} \mathcal{H}_0(y)\, dy,$$

so that

$$t^{\frac{p}{2}-\frac{1}{2}} \| \omega(\cdot,t) \|_p^p = \Big(\frac{|m|}{\sqrt{4\pi c t}} \Big)^p \sqrt{4c} \int_{-\infty}^{+\infty} \frac{1}{|\mu - \sigma h \,\mathsf{erf}(\xi)|^p} \left| \int_0^1 \mathrm{e}^{-\left(\xi + \frac{\alpha - y}{\sqrt{4ct}}\right)^2} \mathcal{H}_0(y)\, dy \right|^p d\xi.$$

Since, for all $\xi \in \mathbf{R}$ and $t \ge 1/4c$, we have

$$\left| \int_0^1 \mathrm{e}^{-\left(\xi + \frac{\alpha - y}{\sqrt{4ct}}\right)^2} \mathcal{H}_0(y)\, dy \right|^p \le \mathrm{e}^{-\frac{p}{2}\xi^2 + 1} \| \mathcal{H}_0 \|_{L^1(0,1)}^p,$$

we get, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \| \omega(\cdot,t) \|_p$$
$$= \frac{|m|}{\sqrt{4\pi c t}} (4c)^{\frac{1}{2p}} \Big(\int_0^1 \mathcal{H}_0(y)\, dy \Big) \left(\int_{-\infty}^{+\infty} \left| \frac{\mathrm{e}^{-\xi^2}}{\mu - \sigma h \,\mathsf{erf}(\xi)} \right|^p d\xi \right)^{1/p}$$
$$= \frac{|m|}{\sqrt{4\pi c t}} (4c)^{\frac{1}{2p}} \left| \frac{2c}{bm} \Big(1 - \mathrm{e}^{-\frac{bm}{2c}} \Big) \right| \| \mathcal{F} \|_p$$

in view of (37), (39). This shows (ii). Finally, for $p = \infty$, we observe that, letting $t \to \infty$ in (45), we get

$$\liminf_{t \to \infty} \; t^{\frac{1}{2}} \, \| \, \omega(\cdot, t) \, \|_\infty \;\geq\; \frac{|\, m \,|}{\sqrt{4 \pi c t}} \left| \, \frac{2 c}{b m} \left(1 - \mathsf{e}^{-\frac{b m}{2 c}} \right) \right| \; \frac{\mathsf{e}^{-\xi^2}}{\mu - \sigma h \, \mathsf{erf}\,(\xi)}$$

for every $\xi \in \mathbf{R}$, so that

$$\liminf_{t \to \infty} \; t^{\frac{1}{2}} \, \| \, \omega(\cdot, t) \, \|_\infty \;\geq\; \frac{|\, m \,|}{\sqrt{4 \pi c t}} \left| \, \frac{2 c}{b m} \left(1 - \mathsf{e}^{-\frac{b m}{2 c}} \right) \right| \; \| \, \mathcal{F} \, \|_\infty . \tag{46}$$

On the other hand, for $t > 0$ let $\xi_t \in \mathbf{R}$ be such that

$$\| \, \omega(\cdot, t) \, \|_\infty \;=\; |\, \omega(\, \alpha + \xi_t \sqrt{4 c t} \, , \, t \,) \,| \, ; \tag{47}$$

since $\liminf_{t \to \infty} \; t^{\frac{1}{2}} \, \| \, \omega(\cdot, t) \, \|_\infty > 0$ from (46), we must have $\xi_t = O(1)$ as $t \to \infty$. Now, given any sequence $t_n \to \infty$ such that $\xi_n \equiv \xi_{t_n}$ converges, say $\xi_n \to \xi_*$, we then have, from (45), (47),

$$\sqrt{t_n} \; \| \, \omega(\cdot, t_n) \, \|_\infty \;=\; \frac{|\, m \,|}{\sqrt{4 \pi c t}} \; \frac{1}{\mu - \sigma h \, \mathsf{erf}\,(\xi_n)} \int_0^1 \mathsf{e}^{-\left(\xi_n + \frac{\alpha - y}{\sqrt{4 c t_n}} \right)^2} \; \mathcal{H}_0(y) \; dy,$$

so that, letting $n \to \infty$, we obtain

$$\lim_{n \to \infty} \sqrt{t_n} \; \| \, \omega(\cdot, t_n) \, \|_\infty \;=\; \frac{|\, m \,|}{\sqrt{4 \pi c t}} \left| \, \frac{2 c}{b m} \left(1 - \mathsf{e}^{-\frac{b m}{2 c}} \right) \right| \; \frac{\mathsf{e}^{-\xi_*^2}}{\mu - \sigma h \, \mathsf{erf}\,(\xi_*)} .$$

This gives

$$\limsup_{t \to \infty} \; t^{\frac{1}{2}} \, \| \, \omega(\cdot, t) \, \|_\infty \;\leq\; \frac{|\, m \,|}{\sqrt{4 \pi c t}} \left| \, \frac{2 c}{b m} \left(1 - \mathsf{e}^{-\frac{b m}{2 c}} \right) \right| \; \| \, \mathcal{F} \, \|_\infty , \tag{48}$$

which, together with (47) above, shows (iii). $\square$

One consequence from Theorem 2 which is worth mentioning it explicitly is the following one.

Theorem 3
Let $a, \, b, \, c, \, \hat{a}, \, \hat{b}, \, \hat{c}$ *be real constants, with* $c, \, \hat{c} > 0$, *and let* $u(x,t), \, \hat{u}(x,t)$ *be the solutions of equations* (17), (18), *respectively, corresponding to initial states* $u_0, \, \hat{u}_0$ *in* $L^1(\mathbf{R})$ *with the same mass* $m \neq 0$. *Then the following statements are equivalent to one another:*

(i) $\quad (\, a, \, b, \, c \,) \;=\; (\, \hat{a}, \, \hat{b}, \, \hat{c} \,)$,

(ii) $\liminf_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p = 0$ *for some* $1 \le p \le \infty$,

(iii) $\lim_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p = 0$ *for all* $1 \le p \le \infty$.

□ Using Lemma 3, it is sufficient to examine the case $u_0 = v_0 = m\,\chi_{[0,1]}$, $m \neq 0$. If $a \neq \hat{a}$, then, from (6), (7), (8), there exist constants $K,\, k > 0$ such that

$$| \, u(\xi\sqrt{t} + \hat{a}\,t,\, t) \, | \;\le\; \frac{K}{\sqrt{t}}\, \mathrm{e}^{-\frac{1}{2}(a - \hat{a})^2\, t}$$

and

$$| \, \hat{u}(\xi\sqrt{t} + \hat{a}\,t,\, t) \, | \;\ge\; \frac{k}{\sqrt{t}}$$

for all $|\,\xi\,| \le 1$ and $t \ge 1$. This clearly gives

$$\| u(\cdot,t) - \hat{u}(\cdot,t) \|_p \;\ge\; \kappa\, t^{-\frac{1}{2}(1-\frac{1}{p})}$$

for all t large and $1 \le p \le \infty$, for some constant $\kappa > 0$.

Assuming now that $a = \hat{a}$, suppose we have $(\,b,\, c\,) \neq (\,\hat{b},\, \hat{c}\,)$: from (11), (16), we can find $1 < p_* < \infty$ such that the limits (1) corresponding to $u(x,t)$ and $\hat{u}(x,t)$ are different, i.e., $\gamma_{p_*} \neq \hat{\gamma}_{p_*}$, where, for every p,

$$\gamma_p \;=\; \lim_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| u(\cdot,t) \|_p, \qquad \hat{\gamma}_p \;=\; \lim_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| \hat{u}(\cdot,t) \|_p.$$

In particular, we get

(49a) $$\liminf_{t\to\infty} \; t^{\frac{1}{2}\left(1-\frac{1}{p_*}\right)} \, \| u(\cdot,t) - \hat{u}(\cdot,t) \|_{p_*} \;\ge\; | \, \gamma_{p_*} - \hat{\gamma}_{p_*} \, | \;>\; 0.$$

Given $p > p_*$, we have, by interpolation of $\|\cdot\|_1$, $\|\cdot\|_p$ at p_*,

$$\| u(\cdot,t) - \hat{u}(\cdot,t) \|_{p_*} \;\le\; \| u(\cdot,t) - \hat{u}(\cdot,t) \|_1^{\frac{p-p_*}{p-1}\,\frac{1}{p_*}} \; \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p^{\frac{p_*-1}{p-1}\,\frac{p}{p_*}}$$

for every $t > 0$. This gives, from (49a) above,

(49b) $$\liminf_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p \;\ge\; C \, | \, \gamma_{p_*} - \hat{\gamma}_{p_*} \, |^{\left(1-\frac{1}{p}\right)\frac{p_*}{p_*-1}},$$

where $C = (\,\gamma_1 + \hat{\gamma}_1\,)^{-\left(1-\frac{p_*}{p}\right)\frac{1}{p_*-1}}$. Similarly, for $p < p_*$, we get

$$\| u(\cdot,t) - \hat{u}(\cdot,t) \|_{p_*} \;\le\; \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p^{\frac{p}{p_*}} \; \| u(\cdot,t) - \hat{u}(\cdot,t) \|_\infty^{1-\frac{p}{p_*}},$$

which gives, using (49a),

$$\text{(49c)} \quad \liminf_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p \;\geq\; | \gamma_{p_*} - \hat{\gamma}_{p_*} |^{\frac{p_*}{p}} \, (\gamma_\infty + \hat{\gamma}_\infty)^{-\frac{p_* - p}{p}}.$$

Hence, in all cases above, $(a, b, c) \neq (\hat{a}, \hat{b}, \hat{c})$ gives, for every $1 \leq p \leq \infty$,

$$\liminf_{t\to\infty} \; t^{\frac{1}{2}(1-\frac{1}{p})} \, \| u(\cdot,t) - \hat{u}(\cdot,t) \|_p \; > \; 0,$$

which, together with Theorem 2, finishes the argument. □

In a similar way, we can show that, given initial states u_0, $\tilde{u}_0 \in L^1(\mathbf{R})$ with different masses, the corresponding solutions $u(\cdot,t)$, $\tilde{u}(\cdot,t)$ of equation (2) satisfy, for every $1 \leq p \leq \infty$,

$$\text{(50)} \qquad \| u(\cdot,t) - \tilde{u}(\cdot,t) \|_p \;\geq\; c_p \, t^{-\frac{1}{2}(1-\frac{1}{p})}$$

for all t large, where c_p is some positive constant.

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